INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS, VOL. 25, 393–405 (1997)

COMPARISON OF PRESSURE CORRECTION SMOOTHERS FOR MULTIGRID SOLUTION OF INCOMPRESSIBLE FLOW

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SUMMARY

We compare the performance of different pressure correction algorithms used as basic solvers in a multigrid method for the solution of the incompressible Navier–Stokes equations on non-staggered grids. Numerical tests were performed on several cases of lid-driven cavity flow using four different pressure correction schemes, including the traditional SIMPLE and SIMPLEC methods as well as novel variants, and varying combinations of underrelaxation parameters. The results show that three of the four algorithms tested are robust smoothers for the multigrid solver and that one of the new methods converges fastest in most of the tests. © 1997 by John Wiley & Sons, Ltd.

Int. J. Numer. Meth. Fluids, **25**: 393–405 (1997).

No. of Figures: 3. No. of Tables: 5. No. of References: 30.

KEY WORDS: multigrid methods; pressure correction methods; incompressible Navier–Stokes equations

1. INTRODUCTION

What are the relative merits of different pressure correction methods used as basic solvers (or smoothers) in multigrid methods for incompressible flow? Multigrid methods have been used to accelerate the convergence of pressure correction methods like SIMPLE in both staggered[1,2] and non-staggered[3–6] discretizations. A comparison of different smoothers, including some pressure correction methods, on a staggered grid was given in Reference 7. We performed a systematic comparison of the convergence of multigrid iterations on a non-staggered grid. Four different pressure correction methods were tested, both well-established methods like SIMPLE[8] and SIMPLEC,[9] and novel variants. The tests comprised different cases of lid-driven cavity flow. Preliminary results of this study were presented in Reference 10.

All this said, what is the purpose of doing such a study for pressure correction methods when coupled smoothers appear to give much better multigrid performance? The answer is versatility: pressure correction methods have been adapted to and used in a wide range of problems of practical interest. We believe that these methods will continue to be the workhorse of applied CFD for still some time to come, a view that is reinforced by the recent publication of a book[11] on the methodology.

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Contract grant sponsor: Research Council of Norway; Contract grant number: 100556/410;
Contract grant number: STP-30074.

CCC 0271–2091/97/040393–13$17.50

Received July 1996
Revised October 1996

The organization of the paper is as follows. In Section 2 we give the cell-centred co-located discretization. In Section 3 we describe the numerical method. We give a unified definition of the pressure correction methods we employ and discuss various aspects of the non-linear multigrid method. In Section 4 we present the results of the calculations. The majority of the results we present is from two-dimensional simulations with the standard methods. We also present three-dimensional results and discuss some algorithmic variants.

2. GOVERNING EQUATIONS AND DISCRETIZATION

The flow of Newtonian fluids is governed by the Navier–Stokes equations, which express conservation of mass and momentum. In the case of steady, incompressible, laminar and isothermal flow these are

$$\nabla \cdot \mathbf{u} = 0, \tag{1}$$

$$\nabla \cdot (\rho \mathbf{u} u_l) = -\frac{\partial p}{\partial x_l} + \mu \nabla^2 u_l + f_l, \tag{2}$$

where ρ is the density, $\mathbf{u} = (u_1, u_2, u_3) = (u, v, w)$ represents the velocity field, p is the pressure and μ is the dynamic viscosity of the fluid. External forces and other source terms are denoted by f.

These equations are discretized on a structured, non-orthogonal grid with primary variables—Cartesian velocity components and pressure—stored in the cell centres. We use a finite volume discretization, integrating Equations (1) and (2) over a general non-orthogonal control volume. If we integrate the continuity equation (1) over a cell and use the Gauss divergence theorem, we get

$$\sum_n \Delta^n (A\mathbf{n} \cdot \mathbf{u}) = 0, \tag{3}$$

where A and $\mathbf{n}$ are the area and normal vector of the cell face respectively. Δ denotes the difference between cell faces, e.g. $\Delta^1 \phi = \phi^{\mathrm{e}} - \phi^{\mathrm{w}}$. In the same way we can write the u_l-momentum equation as

$$\sum_n \Delta^n [A\mathbf{n} \cdot (\rho \mathbf{u} u_l - \mu G u_l)] = -V(G_l p)^{\mathrm{P}} + V f_l^{\mathrm{P}}, \tag{4}$$

where V is the volume of the control volume and G is the discretized gradient operator. The advective term u_l^{e} is discretized either by a first-order upwind difference or by higher-order differencing. The high-order differencing is implemented by defect correction to the first-order scheme. The defect correction approach improves the stability of the iterations and simplifies the implementation of limiter functions to avoid spurious oscillations in the computed solution.

Because the grid is non-orthogonal, the derivatives that occur in the viscous and pressure terms must be evaluated in the transformed curvilinear co-ordinate system. The evaluation of the gradient of u_l in the diffusive term will in general involve all the neighbouring cells. If the grid is orthogonal, the viscous term will, however, only contain contributions from the main co-ordinate directions. We therefore split the viscous flux into a primary flux, containing the orthogonal terms, that we treat implicitly and a secondary (non-orthogonal) flux that we treat explicitly.

In the pressure term the differentiation with respect to the ξ_n-direction is performed by 2δ-centred differences. This will lead to a loss of ellipticity on the grid scale[12] and it is well known that checkerboard oscillations in pressure can occur. To counter this, we use a pressure-weighted interpolation of the cell face velocities in the discretized continuity equation (3). The idea goes back to Rhie and Chow.[13] This interpolation acts as fourth-order stabilizing dispersive term that eliminates the spurious pressure modes.

3. MULTRIGRID METHOD

Unlike other iterative methods, multigrid algorithms offer convergence rates that are independent of the number of grid points. This is achieved by constructing a method that consists of two complementary components. High-frequency errors are efficiently reduced by a *smoother*, while a *coarse grid correction* employing a hierarchy of coarser grids eliminates slow error components.

Pressure correction methods (such as SIMPLE) have been used as single-grid solvers for incompressible fluid flow. These methods show typical smoothing behaviour: convergence is reasonably fast for the first few iterations but then slows down as long-wavelength components dominate the error.

To construct the sequence of grids on the coarse levels, we use cell-centred coarsening, i.e. we lump two-and-two fine grid cells in each direction such that a two-dimensional coarse grid cell comprises four fine grid cells. The coarse grid equation in the non-linear FAS multigrid method[14] can be written in symbolic form as

$$L_c(\bar{\phi}) = L_c(\tilde{\phi}) + \tilde{r}, \tag{5}$$

where ϕ is the vector of dependent variables, a tilde denotes values restricted from the fine grid, $\tilde{\phi} = R\phi$, and an overbar, $\bar{\phi}$, denotes the solution to the coarse grid problem. The fine grid residual is defined by $r = b - L_f(\phi)$. Fine and coarse grid operators L_f and L_c are given by applying the discretization (3), (4) on the fine and coarse grids respectively. The correction term that will be prelongated back to the fine grid is then given by $e = \bar{\phi} - \tilde{\phi}$.

Because the equations are linear in pressure, several previous authors[3–5] have treated pressure in a linear fashion and formulated coarse grid equations directly in terms of a correction for the pressure. We have chosen to follow Johansson and Davidson[6] and apply the non-linear strategy to all terms in the equations.

We did not use nested iteration (full multigrid). This method can give a significant speed-up to the multigrid convergence, but at the cost of increased complexity both in the multigrid cycling and in the need to keep a set of more accurate interpolation operators for the nested iteration phase of the iterations. We therefore chose to restrict the scope of this study to investigate the performance of the different pressure correction methods in the basic multigrid algorithm.

3.1. Pressure correction method

This subsection describes the smoother or basic solver used in the multigrid method. We use a pressure correction method, i.e. the momentum and continuity equations are solved sequentially instead of in a coupled manner. Although these methods are similar to the fractional step or projection methods that were introduced by Chorin[15] and Temam[16] in the late 1960s, the proto-pressure correction algorithm is SIMPLE introduced by Patankar and Spalding[8] and described thoroughly in Reference 17. An iteration of the pressure correction method starts by solving the discretized momentum equations (4) with the best available estimate for the pressure, p^*. The resulting velocity field $\mathbf{u}^*$ will in general not be mass-conserving, and to obtain a velocity field that satisfies the continuity equation (1), we introduce corrections to the velocities and pressure:

$$\mathbf{u} = \mathbf{u}^* + \mathbf{u}', \tag{6}$$

$$p = p^* + p'. \tag{7}$$

To define different pressure correction methods, we prescribe the relation between the velocity corrections and the pressure corrections. The methods we consider can all be described by the expression

$$u_l'^{\mathrm{P}} = \frac{\sum_n d_n^{\mathrm{P}} \delta_2^n p' - \tau r_l^{*\mathrm{P}}}{a^{\mathrm{P}} - \sigma \sum_{\mathrm{nb}} a^{\mathrm{nb}}}, \tag{8}$$

where δ_2^n is a 2δ-centred difference in the ξ_n-direction, a^{P}, a^{nb} and d_n^{P} are coefficients that we take from the linearized, discrete momentum equation and r_l^* denotes the momentum residual. The algorithmic parameters σ and τ can take values of zero and unity. Different choices of these two parameters correspond to different simplifying assumptions made in the derivation of the relation between velocity corrections and pressure corrections. In all the methods we only consider pressure contributions to the velocity corrections such that terms in the form $\sum a^{\mathrm{nb}} u'^{\mathrm{nb}}$ are neglected. We have the following four methods.

1. $\sigma = 0, \tau = 0$ give the standard SIMPLE algorithm of Patankar and Spalding.[8] Here it is assumed that the intermediate momentum equations are solved exactly.
2. $\sigma = 1, \tau = 0$ give the SIMPLEC method introduced by Van Doormal and Raithby.[9] The difference from SIMPLE is that the term $\sum a^{\mathrm{nb}} u'^{\mathrm{P}}$ is subtracted from both sides of the discretized momentum equations. In this way the neglected contributions from the neighbouring cells are reduced, and the authors claim that this eliminates the need for underrelaxation of the pressure.
3. $\sigma = 0, \tau = 1$ give a method that was proposed by Shaw and Sivaloganathan.[18] This method takes into account that the momentum equations are only solved approximately and uses the residual of the incompletely converged equations in the expression for the velocity corrections. This method has a better theoretical smoothing rate (i.e. the high-frequency error reduction that is important in multigrid) but has to our knowledge never been tested. We will call this variant SIMPLESSE.
4. $\sigma = 1, \tau = 1$ give a combination of SIMPLEC and the method proposed in Reference 18 (SIMPLESSEC).

To obtain an equation for the pressure corrections, we interpolate the corrected velocity given by equations (6) and (8) to the cell faces with the pressure-weighted interpolation and insert it into the discretized continuity equation (3). The pressure correction equation will in general have contributions from a full stencil (nine points in 2D, 19 points in 3D). Since the equation describes a correction that vanishes in the final, converged solution, we neglect the terms that correspond to cross-derivatives.

3.2. *Intergrid transfer operators*

In this subsection we describe the intergrid transfer operators, prolongation and restriction, that we use to transmit information between the different grid levels. Prolongation is based on an interpolation rule from the coarse to the fine grid points. The fine-to-coarse restriction can be defined by the average

$$\tilde{u}_i = (Ru)_i = \sum_j R(i,j) u_{2i+j}, \tag{9}$$

where i, j represent multi-indices and the non-zero averaging weights are known as the *stencil* of the restriction. Restriction can also be defined as the adjoint of a prolongation operator and it is

convenient to represent both transfer operators in terms of restriction stencils even though we might use different interpolation rules for the two. We refer to Wesseling[20] for a detailed description of the stencil notation.

In order to ensure multigrid convergence, i.e. for the rate of convergence to be independent of the grid size, the restriction and prolongation must satisfy the accuracy requirement

$$m_R + m_P > M, \tag{10}$$

where m_R is the order of the restriction, m_P is the order of the prolongation and M is the order of the differential operator.[21–23] An interpolation rule is of order m if it interpolates polynomials of order $m-1$ exactly. For the Navier–Stokes equations we have $M=2$ and we will consider transfers based on piecewise constant interpolation ($m=1$) and on linear interpolation ($m=2$). The discretization given by equations (3) and (4) is valid for a non-orthogonal, structured grid. In order to reduce the complexity of the multigrid method, we keep the grid information from the physical space in the discrete operator only and we perform the intergrid transfer operations on equidistant grids on the transformed computational space. The stencil for two-dimensional restriction or prolongation based on piecewise constant interpolation is given by

$$R^1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{11}$$

and in the case of bilinear interpolation we have

$$R^2 = \frac{1}{16}\begin{bmatrix} 1 & 3 & 3 & 1 \\ 3 & 9 & 9 & 3 \\ 3 & 9 & 9 & 3 \\ 1 & 3 & 3 & 1 \end{bmatrix} \tag{12}$$

for cells away from the boundary. In boundary cells the bilinear interpolation stencil must be modified. For the velocities we assume a zero value at the boundary (Dirichlet condition), while for the pressure we assume a zero normal derivative at the boundary (Neumann condition).

3.3. Defect correction for higher accuracy

If we consider the discretized momentum equation (4), the balance of advective fluxes in the ξ_1-direction can be written as

$$F_{\mathrm{e}} u_l^{\mathrm{e}} - F_{\mathrm{w}} u_l^{\mathrm{w}},$$

where $F = A\mathbf{n} \cdot \rho\mathbf{u}$ is the mass flow. The crucial point in the discretization of the advection term is how to choose the u_l-values at the cell faces.

The first-order upstream extrapolation is the only linear scheme that is positive, but it is unacceptably diffusive and we consider higher-order schemes. Let us assume that the advecting flux F is positive. In the transformed computational space, where the grid spacing is uniform, we define a family of higher-order scheme (the κ-schemes) by the first-order upwind value plus a correction term:

$$u_l^{\mathrm{e}} = u_l^{\mathrm{P}} + \frac{1+\kappa}{4}(u_l^{\mathrm{E}} - u_l^{\mathrm{P}}) + \frac{1-\kappa}{4}(u_l^{\mathrm{P}} - u_l^{\mathrm{W}}), \quad -1 \leqslant \kappa \leqslant 1. \tag{13}$$

In this family we find both the linear upwind extrapolation ($\kappa = -1$) and the central scheme ($\kappa = 1$). All the members of this family are second-order-accurate in space and the $\kappa = \frac{1}{3}$ scheme is third-order-accurate for steady advection.

The defect correction iteration for the advective terms can be defined by

$$u_l^{\mathrm{e}} = (u_l^{\mathrm{P}})^{(\mathrm{new})} + [\tfrac{1}{2}\Psi(s_{\mathrm{e}})(u_l^{\mathrm{P}} - u_l^{\mathrm{W}})]^{(\mathrm{old})}, \tag{14}$$

where we have rewritten the scheme in terms of a limiter function Ψ to maintain positivity and the slope ratio $s_{\mathrm{e}} = (u_l^{\mathrm{E}} - u_l^{\mathrm{P}})/(u_l^{\mathrm{P}} - u_l^{\mathrm{W}})$ acts as a trigger to detect situations in which wiggles can occur. This explicit treatment of the correction term shows the defect correction method is essentially a modification of the source terms of the first-order method. We use a limiter proposed by Koren[24,25] that is consistent with the $\kappa = \frac{1}{3}$ scheme:

$$\Psi(s) = \max\{0, \min[2s, \min(\tfrac{1}{3} + \tfrac{2}{3}s, 2)]\}. \tag{15}$$

This limiter maximizes the s-interval where the limiter is switched off. To implement the defect correction iterations, we apply the update (14) in the smoothing step (i.e. for each pressure correction iteration at all levels) as suggested by Heinrichs.[26] We found this approach to be more efficient than either using defect correction as an outer iteration, or inside the multigrid cycle by calculating the residuals based on the high-order discretization.

3.4. Residual overweighting of characteristic error components

In advection–diffusion (flow) problems with a strong characteristic direction, some smooth error components cannot be well approximated on the coarse grid. Such poor coarse grid representation can cause a slow-down of the convergence of the multigrid iterations. This situation was analysed by Brandt and Yavneh,[27] who found that the rate of convergence can be improved by overweighting the residuals transferred to the coarse grid. The coarse grid FAS equation (5) can then be written as

$$L_{\mathrm{c}}(\bar{\phi}) = L_{\mathrm{c}}(\tilde{\phi}) + \eta\tilde{r}, \tag{16}$$

where η is the overweighting factor. The optimal value for η was found to be $\eta = \frac{4}{3}$ for two-level cycles and $\eta \approx \sqrt{2}$ for multilevel W-cycles.

It is important to take into account that the analysis was performed for the limiting case $Re \to \infty$. Furthermore, because of the overweighting, smooth error components that are well represented on the coarse grid will be overcorrected. To avoid the convergence deteriorating because of this overcorrection, we only apply residual overweighting (ROW) in regions with strong advection. That is, if the local Reynolds number is greater than some threshold

$$Re_h = \frac{|\tilde{\mathbf{u}}|l}{\nu} \geqslant Re_{\mathrm{crit}} \tag{17}$$

in a coarse cell, then we set $\eta > 1$, otherwise we set $\eta = 1$.

4. NUMERICAL RESULTS

We have computed several cases of lid-driven cavity flow at different Reynolds numbers using V-cycles with two pressure correction interations for both pre-smoothing and post-smoothing, i.e. a V(2,2) cycle. On the coarsest level we used five iterations. In each pressure correction iteration we employ one alternating line Gauss–Seidel sweep for each of the momentum equations and five sweeps for the pressure correction equation. Unless otherwise stated, the restriction operator is based on piecewise constant interpolation given by the stencil (11). Bilinear interpolation (12) is then used for the coarse-to-fine prolongation.

Iterations were carried out until the residual norm was reduced by a factor $\varepsilon = 10^{-3}$. For the calculations with the first-order upwind discretization this is sufficient to obtain a converged solution.

For the more accurate defect correction iterations the solution is not fully converged, but the difference from the converged solution is so small that it is negligible. To compare the performance of the various pressure correction methods, we consider the average residual reduction factor after n iterations given by

$$\lambda = \left(\frac{\|r_{\text{tot}}^n\|}{\|r_{\text{tot}}^0\|} \right)^{1/n}, \tag{18}$$

where the residual norm $\|r_{\text{tot}}\|$ is the L_2-norm of the combined mass and momentum residuals. We take a work unit to be the cost of performing one iteration with the pressure correction iteration on the fine grid. Because the smoother is relatively expensive, we can neglect the cost of the grid transfer operators. We also neglect the difference in cost between the different pressure correction methods given by Equation (8). The cost of a V(2,2) cycle is thus $4 \times 2^d/(2^d - 1)$ work units, where d is the number of space dimensions.

4.1. Two-dimensional lid-driven cavity

The first set of test cases comprises shear-driven flow at different Reynolds numbers in square, rectangular and skew cavities. In all these tests we used a 64×64 fine grid and four levels in the multigrid iterations, such that the coarsest grid had 8×8 cells. We compared the convergence of the four pressure correction methods with varying relaxation parameters. The test matrix for the entire exercise is given in Table I. As expected, we found that underrelaxation of the momentum equations is crucial to obtain convergence. For single-grid iterations with the SIMPLE method the optimal combination of relaxation parameters is around 0·7–0·8 for velocity and 0·2–0·3 for pressure.[28] The results show that this also seems to be the case for the multigrid iterations. Note also that, in accordance with Reference 9, we obtain the best convergence for the SIMPLEC method without underrelaxation of the pressure.

The converged solution for the square cavity is in good agreement with the benchmark results by Ghia *et al.*,[29] as we can see from the computed centreline velocities for $Re = 1000$ shown in Figure 1.

We performed the calculations with a rectangular and a skew cavity to evaluate the performance of the method for grids with aspect ratios different from unity and for non-orthogonal grids. The rectangular cavity had a 4:1 aspect ratio and the Reynolds numbers were calculated based on the cavity height. In the skew cavity the grid lines intersect at an angle of 60°.

We summarize the results in Table II–IV. The tables give, for each Reynolds number and smoother, an optimal combination of relaxation factors and the convergence factors for the first-order, defect correction and ROW calculations. The optimal relaxation factors were chosen as the combination that gave the fastest convergence. In a few cases the optimal combination was different for the first-order and the higher-order discretizations. In these cases we give priority to the more accurate defect correction calculations. The performance of the methods is, by and large, comparable

Table I. Test matrix for two-dimensional calculations

Geometry	*Re*	Smoother	Relax(u)	Relax(p)	
Square	100	SIMPLE	0·25	0·25	First-order upwind
Rectangular	400	SIMPLEC	0·50	0·50	$\kappa = \frac{1}{3}$ scheme
Skew	1000	SIMPLESSE	0·75	0·75	$\kappa = \frac{1}{3}$ with ROW
		SIMPLESSEC	1·00	1·00	

Total: $3 \times 3 \times 4 \times 4 \times 4 \times 3 = 1728$ combinations.

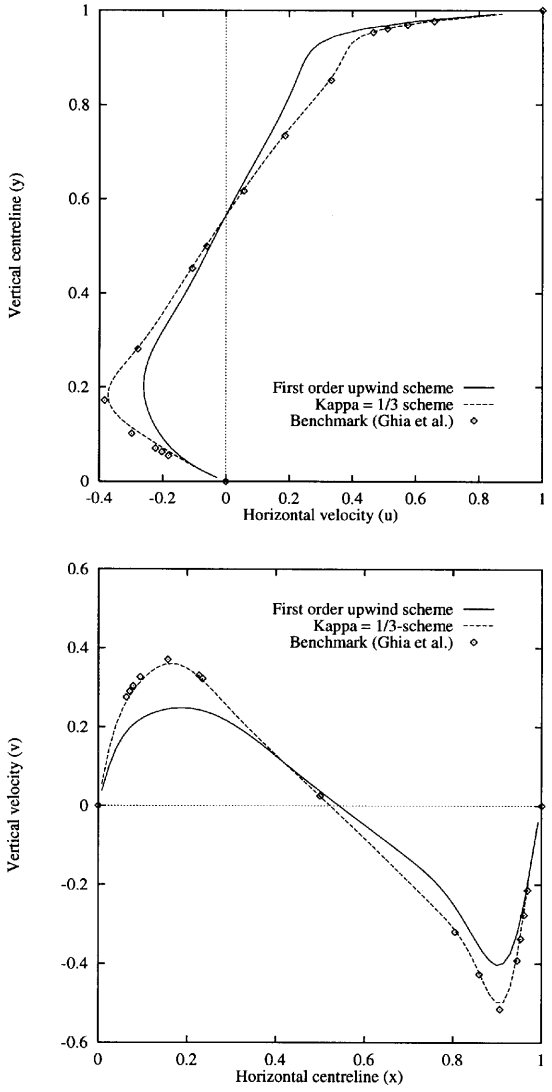

Figure 1. Square cavity. Calculated vertical and horizontal centreline profiles compared with benchmark results of Ghia *et al.*[29]

for each test case. At the higher Reynolds numbers the convergence of the iterations with defect corrections deteriorates compared with the first-order discretizations. Because of the trigger (17), residual overweighting is never applied at the lower Reynolds numbers and these calculations are therefore unaffected by ROW. At $Re = 1000$ the efficiency (measured as $-1/\log\lambda$) can improve by 10%–20%. At even higher Re the impact is, as expected, greater and for $Re = 5000$ the speed-up is about 30%.

The convergence rate in the calculations of the square cavity with SIMPLE and the first-order upwind discretization can be compared with the theoretical analysis of Shaw and Sivaloganathan.[18,19] We obtain convergence factors that are higher than what we expect from the analysis. We do not know the reasons for this discrepancy. The analysis is, however, based on a staggered grid, and the

Table II. Square cavity. Optimal relaxation parameters and convergence factors. First-order upwind differencing, defect corrections with $\kappa = \frac{1}{3}$ scheme and defect corrections accelerated by residual overweighting with $\eta = \frac{4}{3}$ and $Re_{\text{crit}} = 300$

Re	Algorithm	Relax(u, p)	Upwind	DC	ROW
100	SIMPLE	(0·75, 0·25)	0·166	0·167	
	SIMPLEC	(0·75, 1·00)	0·160	0·160	
	SIMPLESSE	(0·75, 0·25)	0·161	0·161	
	SIMPLESSEC	(0·75, 0·50)	0·137	0·137	
400	SIMPLE	(0·75, 0·25)	0·341	0·408	
	SIMPLEC	(0·75, 1·00)	0·343	0·412	
	SIMPLESSE	(0·75, 0·25)	0·292	0·354	
	SIMPLESSEC	(0·75, 0·50)	0·265	0·306	
1000	SIMPLE	(0·75, 0·25)	0·406	0·524	0·463
	SIMPLEC	(0·75, 1·00)	0·408	0·525	0·458
	SIMPLESSE	(0·75, 0·25)	0·360	0·462	0·457
	SIMPLESSEC	(0·50, 0·75)	0·393	0·495	0·629

co-located grid we use changes the ellipticity of the discrete system.[12] It is possible that this affects convergence and can explain the difference.

The results for the rectangular cavity are comparable with the square cavity case, especially at the lower Reynolds numbers. Convergence deteriorates markedly for the skew cavity at high Reynolds numbers. This is probably caused by the neglect of terms corresponding to the cross-derivatives in the pressure correction equation.

The SIMPLE and SIMPLESSE methods show the most consistent results. For these methods the optimal combination of relaxation parameters (0·75, 0·25) gave the fastest convergence in all but one of the cases. We note that the modifications proposed by Shaw and Sivaloganathan[18] do indeed give slightly better convergence than SIMPLE as their analysis suggests. The 'C' variants can give better convergence in some cases, but the optimal choice of relaxation parameters seems to be problem-dependent, especially for the SIMPLESSEC method. We also note that when we apply residual

Table III. Rectangular cavity. Optimal relaxation parameters and convergence factors. First-order upwind differencing, defect corrections with $\kappa = \frac{1}{3}$ scheme and defect corrections accelerated by residual overweighting with $\eta = \frac{4}{3}$ and $Re_{\text{crit}} = 300$

Re	Algorithm	Relax(u, p)	Upwind	DC	ROW
100	SIMPLE	(0·75, 0·25)	0·159	0·171	
	SIMPLEC	(0·75, 1·00)	0·159	0·171	
	SIMPLESSE	(0·75, 0·25)	0·147	0·153	
	SIMPLESSEC	(0·75, 0·50)	0·088	0·093	
400	SIMPLE	(0·75, 0·25)	0·312	0·416	
	SIMPLEC	(0·75, 1·00)	0·314	0·416	
	SIMPLESSE	(0·75, 0·25)	0·277	0·369	
	SIMPLESSEC	(0·50, 0·50)	0·414	0·499	
1000	SIMPLE	(0·75, 0·25)	0·439	0·695	0·663
	SIMPLEC	(0·75, 1·00)	0·441	0·680	0·644
	SIMPLESSE	(0·75, 0·25)	0·381	0·695	0·677
	SIMPLESSEC	(0·50, 0·50)	0·526	0·692	0·663

Table IV. Skew cavity. Optimal relaxation parameters and convergence factors. First-order upwind differencing, defect corrections with $\kappa = \frac{1}{3}$ scheme and defect corrections accelerated by residual overweighting with $\eta = \frac{4}{3}$ and $Re_{\text{crit}} = 100$

Re	Algorithm	Relax(u, p)	Upwind	DC	ROW
100	SIMPLE	(0·50, 0·50)	0·200	0·201	
	SIMPLEC	(0·50, 1·00)	0·200	0·201	
	SIMPLESSE	(0·75, 0·25)	0·231	0·232	
	SIMPLESSEC	(0·75, 0·75)	0·165	0·164	
400	SIMPLE	(0·75, 0·25)	0·350	0·355	
	SIMPLEC	(0·75, 1·00)	0·344	0·349	
	SIMPLESSE	(0·50, 0·25)	0·293	0·296	
	SIMPLESSEC	(0·75, 0·75)	0·245	0·279	
1000	SIMPLE	(0·75, 0·25)	0·627	0·816	0·792
	SIMPLEC	(0·75, 1·00)	0·645	0·819	0·793
	SIMPLESSE	(0·75, 0·25)	0·604	0·781	0·763
	SIMPLESSEC	(0·50, 0·50)	0·585	0·800	0·943

overweighting in combination with the SIMPLESSEC smoother, the convergence actually deteriorates in some cases.

For the residual overweighting we determined the trigger value Re_{crit} by preliminary experiments. Unfortunately, this parameter appears to be problem-dependent as we were not able to find a value that performs well in all cases. This severely restricts the robustness and applicability of the method. ROW is therefore not used in the remainder of this study.

4.2. *Influence of grid transfer operators*

In the examples above and in previous work by other authors,[3–6] prolongation was based on bilinear interpolation, while restriction was based on piecewise constant interpolation. This is a natural choice, especially in the cases where pressure was treated in a linear fashion as in most of those references. In this subsection we will investigate the effect of using different grid transfer operators. Because the transfer operators have to satisfy the accuracy requirement (10), either the restriction or the prolongation must be based on bilinear interpolation. We performed calculations with SIMPLE for the square cavity. Representative results for $Re = 1000$ are given in Figure 2. There is no benefit in swapping the order of the operators or in using higher-order interpolation in both transfers.

4.3. *Three-dimensional lid-driven cavity*

We have also calculated the flow in a three-dimensional cubic cavity. Lid-driven cavity flow is essentially two-dimensional for $Re < 500$ and unstable spanwise modes occur at fairly low Reynolds numbers. This means that only a narrow window of Reynolds numbers provides a test case with steady flow that displays some three-dimensionality. Ramanan and Homsy[30] found that the first unstable mode with a wavelength comparable with the cavity dimension appears at $Re \approx 730$.

We performed single-grid and multigrid iterations on two grids with 32 and 48 cells in each direction. The Reynolds number in the simulations ($Re = 700$) was slightly below the critical value for three-dimensional unstable modes and we used the SIMPLESSE smoother with defect corrections. We did not use residual overweighting because of its lack of robustness discussed above. The convergence histories are shown in Figure 3. Note that the multigrid method performs

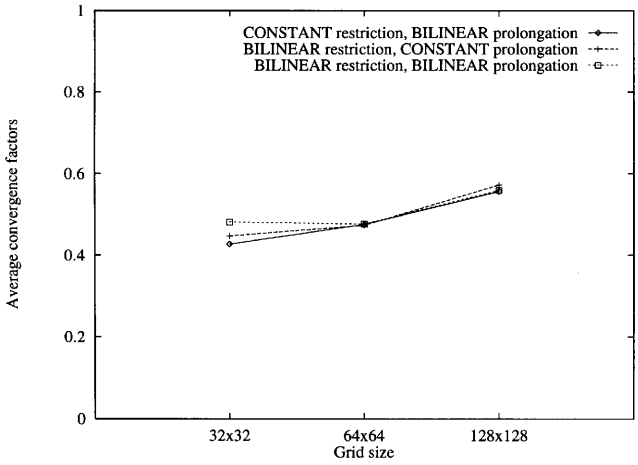

Figure 2. Convergence factors for different choice of transfer operators. Square cavity, $Re = 1000$, SIMPLE

equally well in the two cases and that the convergence of the single-grid iterations deteriorates with grid refinement, as expected. If we take into account the complexity of the multigrid cycle—that one V $(2, 2)$ cycle corresponds to $16/3$ work units (single-grid iterations) in 2D and to $32/7$ work units in 3D—we can calculate the multigrid speed-up factors presented in Table V. It appears that a 50% increase in the number of cells in each direction leads to almost a doubling of the speed-up. The relative cost of a coarse grid correction is smaller in 3D than in 2D; this explains the difference between the 2D and 3D results.

The difference in performance of the pressure correction methods appears to be smaller in 3D than in 2D: SIMPLE, SIMPLEC and SIMPLESSE converged in the same number of iterations. Moreover, the efficiency of the methods (measured by $-1/\log\lambda$) varied by less than 5%, with SIMPLESSE again having the edge over the others. SIMPLESSEC actually converged fastest in this test as in the two-dimensional square cavity calculations. This method is, however, not robust with respect to relaxation factors, as we noted above.

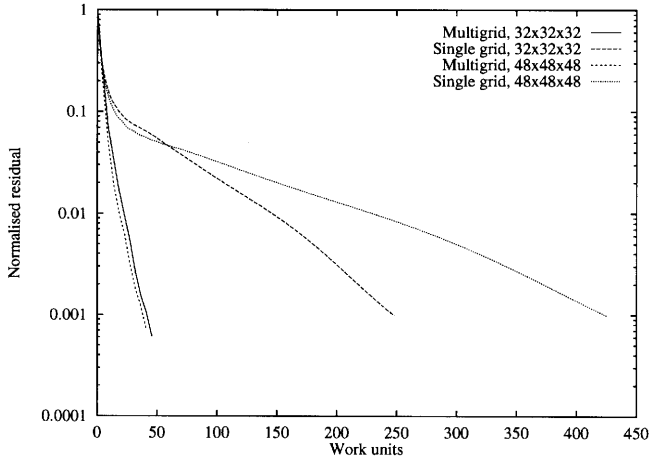

Figure 3. Convergence histories of single-grid and multigrid iterations of lid-driven flow in a cubic cavity. $Re = 700$, SIMPLESSE

Table V. Lid-driven cavity flow, $Re = 700$. Multigrid speed-up factors, SIMPLESSE

Cell size	2D	3D
1/32	5·4	5·8
1/48	9·4	10·8
1/64	14·7	

5. CONCLUSIONS

We compared the convergence of a multigrid solver for incompressible flow with four different pressure correction methods used as smoothers. The number of multigrid iterations needed for convergence is, for practical purposes, independent of the grid resolution. We have observed some small dependence in some cases, but the variations are not systematic as we can see from Figures 2 and 3, where we can observe both marginal slow-down and speed-up as the grids are refined.

There is little difference between the performance of the methods. In particular, the two established algorithms, SIMPLE and SIMPLEC, matched each other case-by-case. One of the new variants, SIMPLESSE, performs in general somewhat better than the two older methods in all the tests, with an improved efficiency of up to 20%. The other novel method, SIMPLESSEC, did shine occasionally for some combination of geometry, Reynolds number and relaxation parameters, only to lose its lustre in the next test.

In general, SIMPLE, SIMPLEC and SIMPLESSE are robust with respect to the underrelaxation factors, in the sense that we found that the optimal relaxation factors were the same in all the tests. Changing the relaxation factors can, however, have a profound effect on the performance of the methods. In particular, the momentum equations should not be underrelaxed too heavily.

We also tried to accelerate the convergence of the coarse grid corrections by residual overweighting. This method gave in some cases a speed-up of about 20% at moderate Reynolds numbers, increasing to 30% for the highest Re we tested. The method is, however, not robust because the trigger value Re_{crit} in (17) is problem-dependent. Overweighting can therefore in general not be recommended as a universally helpful measure.

For the two-dimensional square cavity case with SIMPLE and the first-order upwind discretization there is a difference between the convergence rate we observed in the experiments and the theoretical analysis of Shaw and Sivaloganathan.[18,19] We believe that this difference may be caused by the different grid systems employed in our study and in the analysis.

Changing the restriction and prolongation operators had no significant effect on the convergence of the method. The traditional choice of bilinear prolongation and piecewise constant restriction performed marginally better than the others.

ACKNOWLEDGEMENTS

Ms. Lossius performed part of this work in preparation for her Cand. Scient. degree at the Department of Mathematics, University of Bergen, She thanks Christian Michelsen Research AS (CMR) for their hospitality and support. Mr. Gjesdal was supported by the Research Council of Norway through grant 100556/410. We are grateful for permission to use the multifluid simulation code (MUSIC) developed by CMR with support from the Research Council of Norway through programme STP-30074.

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